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Structure of the three-dimensional quantum euclidean space

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Abstract. As an example of a noncommutative space we discuss the quantum 3-dimensional Euclidean space $\mathbb{R}^3_q$ together with its symmetry structure in great detail. The algebraic structure and the representation theory are clarified and discrete spectra for the coordinates are found. The q-deformed Legendre functions play a special role. A completeness relation is derived for these functions.

1 Introduction

Noncommutative space-time structures are finding increasing interest in gauge theories. Special examples [1] have been discussed in the literature. Among them are

1) The canonical structure:

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij}, \qquad \Theta^{ij} \in \mathbb{C}$$

2) the Lie-algebra structure:

$$[\hat{x}^i, \hat{x}^j] = i\, c^{ij}_k \hat{x}^k, \qquad c^{ij}_k \in \mathbb{C}$$

3) The quantum space structure:

$$[\hat{x}^i, \hat{x}^j] = i c^{ij}_{kl} \hat{x}^k \hat{x}^l, \qquad c^{ij}_{kl} \in \mathbb{C}$$

We shall discuss a special example of the third case [2]. For a more general review of the quantum space structure see [3].

In all of the above cases we consider the associative algebra freely generated by the elements $\hat{x}^i$ modulo the respective relations. This algebra of formal power series forms the algebra $\mathcal{A}_x$.

$$\mathcal{A}_x = \frac{\mathbb{C}[\,[\hat{x}^1 \ldots \hat{x}^N]\,]}{R}$$

For a physicist this means that he is free to use the relations to reorder the elements of an arbitary power series. For the quantum space algebra we wish to exclude pathological cases such as the trivial case where there is no relation at all or where the product of any two elements is zero modulo the relations. To exclude such cases we shall demand the Poincaré-Birkhoff-Witt property for the algebra. By this we mean that the dimension of the space of homogeneous polynomials is the same as in the commutative case. For the first and second examples this will be the case, for the third example we require the Yang-Baxter equation. To formulate it we write the relations in the form:

$$\hat{x}^i \hat{x}^j = \hat{R}^{ij}_{kl} \hat{x}^k \hat{x}^l, \qquad \hat{R}^{ij}_{kl} \in \mathbb{C}$$

and define $N^3 \times N^3$ matrices

$$\begin{aligned} \hat{R}_{12}{}^{i_1\, i_2\, i_3}_{j_1\, j_2\, j_3} &= \hat{R}^{i_1\, i_2}_{j_1\, j_2}\, \delta^{i_3}_{j_3} \\ \hat{R}_{23}{}^{i_1\, i_2\, i_3}_{j_1\, j_2\, j_3} &= \delta^{i_1}_{j_1}\, \hat{R}^{i_2\, i_3}_{j_2\, j_3} \end{aligned}$$

The Yang-Baxter equation is:

$$\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12} \;=\; \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23}$$

There are several known solutions of this equation. We are interested in such relations that allow a conjugation which makes the algebra a $*$-algebra. This is because we have to associate the observables like the coordinates with selfadjoint operators in a Hilbertspace. The $\hat{R}$-matrices for the quantum groups $SO_q(n)$ allow such conjugations.

The quantum space algebra is a comodule of a quantum group. We start from co-algebra relations

$$\Delta(x^i) = T^i_k \otimes \hat{x}^k$$

and compute

$$\begin{aligned} \Delta(x^i)\Delta(x^j) &= T^i_k T^j_l \otimes \hat{x}^k \hat{x}^l \\ &= T^i_k T^j_l \otimes \hat{R}^{kl}_{mn} \hat{x}^m \hat{x}^n \end{aligned}$$

If we demand RTT relations

$$T^i_k T^j_l \hat{R}^{kl}_{mn} = \hat{R}^{ij}_{kl} T^k_m T^l_n$$

for the T-algebra, we find

$$\Delta(x^i)\Delta(x^j) = \hat{R}^{ij}_{kl} \Delta(x^k)\Delta(x^j)$$

There is always a solution to the RTT relations given by

$$T^i_j = \delta^i_j.$$

If this is the only solution then the bialgebra consists of the unit element only; not very interesting. If the $\hat{R}$-matrix leads to the quantum group $SO_q(n)$, we have a more interesting case. Instead of introducing the quantum group $SO_q(n)$ we shall deal with the corresponding q-Lie algebra $so_q(n)$. The quantum space is then a module of this algebra.

In this paper we discuss the 3-dimensional case in great detail. The algebra is introduced in Chapter 2. It has a peculiar property, there is a homomorphism of the algebra $so_q(3)$ into the algebra $\mathbb{R}_q^3$. This is discussed in Chapter 3. The full algebra can then be generated by a central element, the radius R, and elements of the tensor product of an $su_q(2)$ algebra and an $su_q(1,1)$ algebra. The generators of the $su_q(2)$ algebra are further restricted by relations that when the algebra is represented lead to a unique infinite-dimensional representation of $su_q(2)$. We call this algebra the t-algebra. The $su_q(1,1)$ algebra we call K-algebra. If we then demand that the $so_q(3)$ algebra corresponds to orbital angular momentum the K algebra is restricted in the same sense as the t-algebra. This is discussed in Chapter 4. This clarifies the algebraic structure of the $so_q(3)$ module $\mathbb{R}_q^3$.

To discuss physics we need representations of the algebra. The observables should be represented by (essentially) self-adjoint linear operators in a Hilbert space. This way we can use the well developped formalism of quantum mechanics and its interpretation scheme. In Chapter 5 we discuss the representations of the algebra. We find that they are characterized by one real parameter z_0. In all these representations we obtain a discrete spectrum for the coordinate X^3, which along with R and T^3_{orb}, the third component of the orbital angular momentum, form a complete commuting set of observables. The scale of the spacing of the eigenvalues of X^3 is determined by the constant z_0, the eigenvalues are exponentially spaced. This we call a q-lattice. We are not surprised that noncommuting variables lead to a discretization (latticization) of space [4].

In Chapter 6 we construct the transformation that leads to a basis where $\mathbf{T_{orb}}^2$ is diagonal. The corresponding transformation function turn out to be the q-deformed associated Legendre functions. They are defined in Appendix D in terms of the big q-Jacobi polynomials. They satisfy a difference equation, a recursion equation and have orthogonaltity properties - in complete analogy to the usual associated Legendre functions. From the self-adjointness property of X^3 we derive a completeness relation as well, this is done in Appendix E. Appendices A, B and C are devoted to the representation of the $su_q(2)$ and $su_q(1,1)$ algebras and their comultiplication.

2 The algebra of the euclidean quantum space $\mathbb{R}_q^3$

This algebra has been discussed in [5], we use the same notation here: $\mathbb{R}_q^3$:

$$\begin{aligned} X^3X^+ - q^2X^+X^3 &= 0 \\ X^3X^- - q^{-2}X^-X^3 &= 0 \\ X^-X^+ - X^+X^- &= \lambda X^3X^3, \qquad \lambda = q - q^{-1}, \quad q \in \mathbb{R}. \end{aligned} \tag{2.1}$$

We shall assume $q > 1$ in this paper. This non-commutative structure is our model for a non-commutative space. We can impose conjugation properties that are compatible with the relations (2.1) justifying the '$\mathbb{R}$' in $\mathbb{R}_q^3$:

$$\overline{X^+} = -qX^-, \qquad \overline{X^3} = X^3. \tag{2.2}$$

The quantum space $\mathbb{R}_q^3$ has a co-module structure under the action of the quantum group $SO_q(3)$ [3] and a module structure under the corresponding q-Lie algebra. $su_q(2)$:

$$\begin{aligned} q^{-1}T^+T^- - qT^-T^+ &= T^3 \\ q^2T^3T^+ - q^{-2}T^+T^3 &= (q+q^{-1})T^+ \\ q^2T^-T^3 - q^{-2}T^3T^- &= (q+q^{-1})T^- \end{aligned} \tag{2.3}$$

The conjugation properties justifying the 'u' in $su_q(2)$ are:

$$\overline{T^+} = \frac{1}{q^2}T^-, \qquad \overline{T^3} = T^3. \tag{2.4}$$

The module structure that was found in [5] is:

$$\begin{aligned} T^3X^3 &= X^3T^3 \\ T^3X^+ &= q^{-4}X^+T^3 + q^{-1}(1+q^{-2})X^+ \\ T^3X^- &= q^4X^-T^3 - q(1+q^2)X^- \end{aligned} \tag{2.5}$$

$$\begin{aligned} T^+X^3 &= X^3T^+ + q^{-2}\sqrt{1+q^2}X^+ \\ T^+X^+ &= q^{-2}X^+T^+ \\ T^+X^- &= q^2X^-T^+ + q^{-1}\sqrt{1+q^2}X^3 \end{aligned} \tag{2.6}$$

$$\begin{aligned} T^-X^3 &= X^3T^- + q\sqrt{1+q^2}X^- \\ T^-X^+ &= q^{-2}X^+T^- + \sqrt{1+q^2}X^3 \\ T^-X^- &= q^2X^-T^- \end{aligned} \tag{2.7}$$

In the limit $q = 1$ we obtain from relations (2.1)–(2.7) the commutative $\mathbb{R}^3$ with the Lie algebra $so(3)$ acting on it.

As a consequence of the above relations it follows that there is a central hermitean element, the q-deformed radius:

$$\begin{aligned} R^2 &= X^3X^3 - qX^+X^- - \frac{1}{q}X^-X^+ \\ &= q^2\overline{X^3}X^3 + (1+q^{-2})\overline{X^+}X^+, \\ \overline{R^2} &= R^2. \end{aligned} \tag{2.8}$$

'Central' means that R^2 commutes with all the elements X and T.

There is a well-known Casimir operator for the $su_q(2)$ algebra:

$$\mathbf{T}^2 = \frac{q^2}{\lambda^2}\tau^{\frac{1}{2}} + \frac{1}{\lambda^2}\tau^{-\frac{1}{2}} + \tau^{-\frac{1}{2}}T^+T^- - \frac{1+q^2}{\lambda^2}. \tag{2.9}$$

We have introduced the group-like element

$$\tau = 1 - \lambda T^3 \tag{2.10}$$

and the elements $\tau^{\frac{1}{2}}$ and $\tau^{-\frac{1}{2}}$ as an extension of the algebra. We shall extend the algebra by the element $R = (R^2)^{\frac{1}{2}}$ and $R^{-1} = (R^2)^{-\frac{1}{2}}$ as well.

The τX and τT commutation relations can be obtained from the $T^3 X$ and $T^3 T$ relations and vice versa. They are

$$\begin{aligned} \tau X^3 &= X^3 \tau \\ \tau X^+ &= q^{-4} X^+ \tau \\ \tau X^- &= q^4 X^- \tau \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \tau T^3 &= T^3 \tau \\ \tau T^+ &= q^{-4} T^+ \tau \\ \tau T^- &= q^4 T^- \tau . \end{aligned} \tag{2.12}$$

The definition of the orbital angular momentum as it was given in [5] can be best formulated in terms of the elements

$$\begin{aligned} L^+ &= \frac{1}{q^2\sqrt{1+q^2}} \tau^{-\frac{1}{2}} T^+ \\ L^- &= -\frac{1}{q^3\sqrt{1+q^2}} \tau^{-\frac{1}{2}} T^- \\ L^3 &= \frac{1}{q^2(1-q^2)} \left(\tau^{-\frac{1}{2}} - 1 - \frac{\lambda^2}{1+q^2} \mathbf{T}^2 \right) \end{aligned} \tag{2.13}$$

As the q-generalization of the fact that orbital angular momentum is orthogonal to the coordinate vector we impose the constraint

$$L \circ X = L^3 X^3 - q L^+ X^- - \frac{1}{q} L^- X^+ = 0. \tag{2.14}$$

We shall see that this defines orbital angular momentum uniquely.

3 The t algebra

The algebra introduced in the previous chapter allows a homomorphism of the T algebra into the X algebra. This was first seen in [6]. We find this homomorphism by interpreting (2.5), (2.6) and (2.7) as inhomogeneous equations which can be solved for T in terms of X. We first construct a particular solution t and exhibit the homomorphism

$$\begin{aligned} T^+ \mapsto t^+ &= -\frac{1}{\lambda q^3}\sqrt{1+q^2} X^+ (X^3)^{-1} \\ T^- \mapsto t^- &= \frac{q^2}{\lambda}\sqrt{1+q^2} X^- (X^3)^{-1} \\ T^3 \mapsto t^3 &= \frac{1}{\lambda}\left(1 + R^2 (X^3)^{-2}\right). \end{aligned} \tag{3.1}$$

Here we extend the algebra by the inverse of X^3. To establish the homomorphism we have to use (2.1) to show that the t elements satisfy (2.3). Furthermore the relations (2.5) to (2.7) are fullfilled by the t elements. It is due to (2.2) that they satisfy (2.4) as well.

There are additional relations for the t elements that follow from (2.1). They are

$$\tau_t = 1 - \lambda t^3 = -R^2 (X^3)^{-2} \tag{3.2}$$

and

$$\begin{aligned} t^+ t^- &= -\frac{1}{\lambda^2}\left(1 + q^2 \tau_t\right) \\ t^- t^+ &= -\frac{1}{\lambda^2}\left(1 + \frac{1}{q^2}\tau_t\right). \end{aligned} \tag{3.3}$$

It follows that the Casimir operator for the t algebra takes a definite value and that in the notation of Appendix A, where $\overline{m}_t$ and d_t are defined,

$$\mathbf{T}^2 = -\frac{1+q^2}{\lambda^2}, \qquad \overline{m}_t = 0, \qquad d_t = -\frac{q^2}{\lambda}. \tag{3.4}$$

This value of the Casimir operator and the sign of τ_t, which is negative, show that the t algebra cannot be represented by the well-known finite dimensional representations of the T algebra [3]. In Appendix A we shall show that there are infinite-dimensional representations of the T algebra among which there is one satisfying (3.2), (3.3) and (3.4). The representation is uniquely determined by these conditions, we present it here:

$$\begin{aligned} t^3 \ |m_t\rangle &= \frac{1}{\lambda}\left(1 + q^2 q^{-4m_t}\right) \ |m_t\rangle \\ t^+ \ |m_t\rangle &= \frac{1}{\lambda q}\sqrt{q^{-4m_t} - 1} \ |m_t + 1\rangle \\ t^- \ |m_t\rangle &= \frac{q}{\lambda}\sqrt{q^{-4(m_t-1)} - 1} \ |m_t - 1\rangle \\ & m_t \le 0. \end{aligned} \tag{3.5}$$

From (3.5) it follows that

$$t^+ \ |0\rangle = 0. \tag{3.6}$$

There is no state with positive m_t.

Equations (3.1) allow us to express the elements XR^{-1} in terms of the t elements:

$$\begin{aligned} X^3 R^{-1} &= \pm(-\tau_t)^{-\frac{1}{2}} \\ X^+ R^{-1} &= \mp\frac{\lambda q^3}{\sqrt{1+q^2}} t^+ (-\tau_t)^{-\frac{1}{2}} \\ X^- R^{-1} &= \pm\frac{\lambda}{q^2\sqrt{1+q^2}} t^- (-\tau_t)^{-\frac{1}{2}} \end{aligned} \tag{3.7}$$

The two different signs are the signs of $X^3 R^{-1} = \sqrt{(X^3)^2 R^{-2}}$. These elements can be viewed as homogeneous coordinates in the $\mathbb{R}_q^3$ space.

The representations of these elements are now obtained from (3.5):

$$\begin{aligned} X^3R^{-1}\ |m_t\rangle &= \pm q^{2m_t-1}\ |m_t\rangle \\ X^+R^{-1}\ |m_t\rangle &= \mp\frac{q}{\sqrt{1+q^2}}\sqrt{1-q^{4m_t}}\ |m_t+1\rangle \\ X^-R^{-1}\ |m_t\rangle &= \pm\frac{1}{\sqrt{1+q^2}}\sqrt{1-q^{4(m_t-1)}}\ |m_t-1\rangle \end{aligned} \tag{3.8}$$

The different signs in (3.8) lead to inequivalent irreducible representations of the X algebra.

4 The K algebra

We continue to consider the (2.5), (2.6) and (2.7) as inhomogeneous equations that should be solved for the T's. We have found one particular solution (3.1) and now move to the homogeneous part. This we do by the Ansatz:

$$\begin{aligned} T^\pm &= \Delta^\pm + t^\pm \\ T^3 &= \Delta^3 + t^3 \end{aligned} \tag{4.1}$$

(2.5), (2.6) and (2.7) become homogeneous equations for the Δ's.

$$\begin{aligned} X^\pm\Delta^+ &= q^{\pm 2}\Delta^+X^\pm \\ X^3\Delta^+ &= \Delta^+X^3 \end{aligned} \tag{4.2}$$

$$\begin{aligned} X^\pm\Delta^- &= q^{\pm 2}\Delta^-X^\pm \\ X^3\Delta^- &= \Delta^-X^3 \end{aligned} \tag{4.3}$$

$$\begin{aligned} X^\pm\Delta^3 &= q^{\pm 4}\Delta^3X^\pm \\ X^3\Delta^3 &= \Delta^3X^3 \end{aligned} \tag{4.4}$$

These equations suggest the further Ansatz:

$$\begin{aligned} K^\pm &= \pm(-\tau_t)^{-\frac{1}{2}}\Delta^\pm, \\ K^3 &= (\tau_t)^{-1}\Delta^3. \end{aligned} \tag{4.5}$$

The element τ_t satisfies the relation (2.11) and as a consequence all the Ks commute with all the Xs and therefore with all the t's as well

$$\begin{aligned} K^AX^B &= X^BK^A \\ K^At^B &= t^BK^A \end{aligned} \tag{4.6}$$

Now we turn to (2.3) and compute the KK relations:

$$\begin{aligned} q^{-1}K^+K^- - qK^-K^+ &= K^3 \\ q^2K^3K^+ - q^{-2}K^+K^3 &= (q+q^{-1})K^+ \\ -q^{-2}K^3K^- + q^2K^-K^3 &= (q+q^{-1})K^- \end{aligned} \tag{4.7}$$

This is exactly the same algebra as (2.3). Any realization of the K-algebra will lead to a realization of the T-algebra:

$$\begin{aligned} T^\pm &= t^\pm \pm (-\tau_t)^{\frac{1}{2}}K^\pm \\ T^3 &= t^3 + \tau_tK^3 \end{aligned} \tag{4.8}$$

This is a relation which is familiar from the comultiplication of two representations of the algebra (2.3):

$$\begin{aligned} \Delta_\beta(T^3) &= T^3\otimes 1 + \tau\otimes T^3 \\ \Delta_\beta(T^\pm) &= T^\pm\otimes 1 \pm \sqrt{-\tau}\otimes T^\pm \end{aligned} \tag{4.9}$$

This comultiplication will be discussed in Appendix C. It is adjusted to representations where the first factor has negative eigenvalues of τ. We emphasize that the representations of the t algebra in (4.8) are restricted by the relations (3.3) whereas for the K algebra any representation would do as long as we are not considering any conjugation properties.

If we now demand the conjugation property (2.4) for the T algebra we find for the K algebra:

$$\overline{K^3} = K^3, \qquad \overline{K^+} = -\frac{1}{q^2}K^-. \tag{4.10}$$

Note the sign. The K algebra belongs to the $SU_q(1,1)$ quantum group.

If we now use the condition (2.14) for orbital angular momentum we will specify the K algebra representation uniquely as well. It needs some computation to express the L algebra (2.13) in terms of the t and K algebras.

$$\begin{aligned} L^+ &= \frac{1}{q^2\sqrt{1+q^2}}\Big\{(-\tau_t)^{-\frac{1}{2}}t^+\otimes(-\tau_k)^{-\frac{1}{2}} \\ &\quad +1\otimes(-\tau_k)^{-\frac{1}{2}}K^+\Big\} \\ L^- &= -\frac{1}{q^3\sqrt{1+q^2}}\Big\{(-\tau_t)^{-\frac{1}{2}}t^-\otimes(-\tau_k)^{-\frac{1}{2}} \\ &\quad -1\otimes(-\tau_k)^{-\frac{1}{2}}K^-\Big\} \\ L^3 &= \frac{q^2-1}{q^4(q^2+1)}\Big\{\frac{q^2}{\lambda^2}(-\tau_t)^{\frac{1}{2}}\otimes(-\tau_k)^{\frac{1}{2}} \\ &\quad +(-\tau_t)^{\frac{1}{2}}\otimes(-\tau_k)^{-\frac{1}{2}}\left(-K^+K^- + \frac{q^2}{\lambda^2}\right) \\ &\quad -\frac{1+q^2}{\lambda^2}(-\tau_t)^{-\frac{1}{2}}\otimes(-\tau_k)^{-\frac{1}{2}} + t^-\otimes(-\tau_k)^{-\frac{1}{2}}K^+ \\ &\quad -q^2t^+\otimes(-\tau_k)^{-\frac{1}{2}}K^-\Big\} \end{aligned} \tag{4.11}$$

This already shows that we should restrict the representations such that τ_k has negative eigenvalues. There is an additional reason for it. We shall see in Appendix C that the coproduct (4.8) only leads to representations with positive eigenvalues of τ if τ_k has negative eigenvalues. Only in this case the representations of T can be decomposed into finite-dimensional ones. We are here adding this as an additional assumption - not knowing if it is really necessary.

With this assumption it will follow from the comultiplication rule of Appendix C that we have to choose

$$d_k = -\frac{1}{\lambda q^2}. \tag{4.12}$$

Now we are ready to evaluate (2.14). This relation will be true if and only if:

$$\underline{m}_k = -1. \tag{4.13}$$

This is in the notation of Appendix B. For the Casimir operator we find

$$\mathbf{T}_k^2 = -\frac{1+q^2}{\lambda^2}. \tag{4.14}$$

This uniquely determines the K algebra representation. The generators of the orbital angular momentum will be denoted by T_{orb}.

We find the result:

$$\begin{aligned} T^3_{orb} &= t^3 \otimes 1 + \tau_t \otimes K^3 \\ T^{\pm}_{orb} &= t^{\pm} \otimes 1 \pm \sqrt{-\tau_t} \otimes K^{\pm} \end{aligned} \tag{4.15}$$

where the t and K representations are determined by (3.4) and (4.12), (4.13).

We can add spin to orbital angular momentum:

$$\begin{aligned} T^3 &= T^3_{orb} \otimes 1 + \tau_{orb} \otimes S^3 \\ T^{\pm} &= T^{\pm}_{orb} \otimes 1 + \sqrt{\tau_{orb}} \otimes S^{\pm} \end{aligned} \tag{4.16}$$

The spin operators S can be in any finite-dimensional representations of the T algebra.

5 Representations of the T_{orb} algebra

The representation of the K algebra that enters orbital angular momentum is characterized by (4.12), (4.13) and (4.14):

$$d_k = -\frac{1}{\lambda q^2}, \qquad \underline{m}_k = -1, \qquad \mathbf{T}_k^2 = -\frac{1+q^2}{\lambda^2}. \tag{5.1}$$

It is an infinite-dimensional representation with m_k ranging from 0 to ∞.

$$\begin{aligned} K^3 \ |m_k\rangle &= \frac{1}{\lambda}\left(1 + \frac{1}{q^2} q^{-4m_k}\right) \ |m_k\rangle \\ K^+ \ |m_k\rangle &= \frac{1}{q\lambda}\sqrt{1 - q^{-4(m_k+1)}} \ |m_k+1\rangle \\ K^- \ |m_k\rangle &= -\frac{q}{\lambda}\sqrt{1-q^{-4m_k}} \ |m_k - 1\rangle \\ K^- \ |0\rangle &= 0, \qquad m_k \geq 0 \end{aligned} \tag{5.2}$$

The representation T_{orb} of orbital angular momentum is the tensor product of this representation and the t representation given in (3.5). The eigenstates of T^3_{orb} are characterized by the two numbers m_t and m_k.

$$\begin{aligned} T^3_{orb} \ |m_t, m_k\rangle &= \frac{1}{\lambda}\left(1 - q^{-4(m_t+m_k)}\right) \ |m_t, m_k\rangle \\ T^+_{orb} \ |m_t, m_k\rangle &= \frac{1}{\lambda q}\sqrt{q^{-4m_t} - 1} \ |m_t + 1, m_k\rangle \\ &\quad + \frac{1}{\lambda} q^{-2m_t}\sqrt{1 - q^{-4(m_k+1)}} \ |m_t, m_k+1\rangle \\ T^-_{orb} \ |m_t, m_k\rangle &= \frac{q}{\lambda}\sqrt{q^{-4(m_t-1)} - 1} \ |m_t - 1, m_k\rangle \\ &\quad + \frac{q^2}{\lambda} q^{-2m_t}\sqrt{1 - q^{-4m_k}} \ |m_t, m_k - 1\rangle \end{aligned} \tag{5.3}$$

In this representation $\mathbf{K}^2$ and $\mathbf{t}^2$ are diagonal. The transformation to the basis where $\mathbf{T}^2_{orb}$ is diagonal will be constructed in the next chapter. The value of T^3_{orb} in (5.3) shows that we have found finite-dimensional representations of T_{orb}.

We obtain from (5.3)

$$\tau_{orb} \ |m_t, m_k\rangle = q^{-4(m_t+m_k)} \ |m_t, m_k\rangle. \tag{5.4}$$

The representation of the X algebra can be obtained from (3.8). The element R is central, it will be diagonal in the m_t, m_k basis. We denote the eigenvalue of R^2 by

$$R^2 \ |m_t, m_k, M\rangle = q^{4M+2} z_0^2 \ |m_t, m_k, M\rangle, \tag{5.5}$$

where z_0 is an arbitrary parameter characterizing the radius. Then we obtain from (3.8) the representation of X^3:

$$X^3 \ |m_t, m_k, M\rangle = q^{2(m_t+M)} z_0 \ |m_t, m_k, M\rangle. \tag{5.6}$$

We have absorbed the sign in (3.8) yielding inequivalent representations in the sign of z_0 which is not determined by (5.5).

This and (5.4) suggest that we should introduce a notation characterizing the eigenvalue of X^3 by a quantum number as well as the eigenvalue of T^3_{orb}.

$$\nu = m_t + M, \qquad m = m_t + m_k \tag{5.7}$$

In this notation we obtain the representation which was also found in [7]:

$$\begin{aligned} & X^3 \ |M,\nu,m\rangle = q^{2\nu} z_0 \ |M,\nu,m\rangle \\ & R^2 \ |M,\nu,m\rangle = q^{4M+2} z_0^2 \ |M,\nu,m\rangle \\ & T^3_{orb} \ |M,\nu,m\rangle = \frac{1}{\lambda}\left(1 - q^{-4m}\right) \ |M,\nu,m\rangle \\ & X^+ \ |M,\nu,m\rangle = \\ & \quad -\frac{q^2 z_0}{\sqrt{1+q^2}}\sqrt{q^{4M} - q^{4\nu}} \ |M,\nu+1,m+1\rangle \\ & X^- \ |M,\nu,m\rangle = \\ & \quad \frac{q z_0}{\sqrt{1+q^2}}\sqrt{q^{4M} - q^{4(\nu-1)}} \ |M,\nu-1,m-1\rangle \\ & T^+_{orb} \ |M,\nu,m\rangle = \\ & \quad \frac{1}{q^2-1}\sqrt{q^{4(M-\nu)} - 1} \ |M,\nu+1,m+1\rangle \\ & \quad + \frac{1}{\lambda}\sqrt{q^{4(M-\nu)} - q^{-4(m+1)}} \ |M,\nu,m+1\rangle \\ & T^-_{orb} \ |M,\nu,m\rangle = \\ & \quad \frac{q^2}{q^2-1}\sqrt{q^{4(M-\nu+1)} - 1} \ |M,\nu-1,m-1\rangle \\ & \quad + \frac{q^2}{\lambda}\sqrt{q^{4(M-\nu)} - q^{-4m}} \ |M,\nu,m-1\rangle \\ & \nu \leq M, m \geq \nu - M \end{aligned} \tag{5.8}$$

6 Reduction of the representation of T_{orb}

The above representation (5.3) of T_{orb} is a tensor product of two representations, with $\mathbf{t}^2$ and $\mathbf{K}^2$ diagonal. We proceed with its decomposition into a sum of irreducible representations characterized by the eigenvalues of ${\mathbf{T_{orb}}}^2$. From the Appendix A we know that for $d = \lambda d_t d_k = \lambda^{-1}$ the eigenvalues of $\mathbf{T}^2$ are $q[l][l+1]$. Therefore we start with an Ansatz of the form

$$|l,m\rangle = \sum_{m_k,m_t} c_{l,m}^{m_k,m_t} \; |m_t,m_k\rangle, \qquad m_k \geq 0, m_t \leq 0$$
$$\mathbf{T}^2_{orb} \; |l,m\rangle = q[l][l+1] \; |l,m\rangle. \tag{6.1}$$

According to (5.7) $m = m_t + m_k$, so that we have

$$c_{l,m}^{m_k,m_t} = c_{l,m}^{m_t}\delta_{m,m_t+m_k}. \tag{6.2}$$

From the definition (2.9) of $\mathbf{T}^2$ and the Equations (5.3) we obtain a recursion relation for the coefficients $c_{l,m}^{m_t}$.

$$\begin{aligned}&\left(q^{2l+2} + q^{-2l} - (q^2+1)q^{2(m+1)-4m_t}\right) c_{l,m}^{m_t} = \\ &q^{2m+1}\left(\sqrt{(q^{-4m_t}-1)(q^{-4m_t}-q^{-4m})}c_{l,m}^{m_t+1}\right. \\ &\qquad\left.+\sqrt{(q^{4-4m_t}-1)(q^{4-4m_t}-q^{-4m})}c_{l,m}^{m_t-1}\right)\end{aligned} \tag{6.3}$$

A comparison with the q-difference Equation (D.18) for the functions $\widetilde{P}^l_m$ defined in (D.8) and (D.16) shows that (6.3) is solved by

$$c_{l,m}^{m_t} = \begin{cases} \dfrac{\sqrt{1-q^{-2}}}{q^{m+1-m_t}}\widetilde{P}^m_l(\pm q^{2(m_t-1)-2m}) \text{ for} m \geq 0 \\ \dfrac{\sqrt{1-q^{-2}}}{q^{1-m_t}}\widetilde{P}^{|m|}_l(\pm q^{2(m_t-1)}) \quad \text{ for} m < 0 \end{cases} \tag{6.4}$$

Note that P^m_l is defined for $m \geq 0$ only.

The orthogonality condition (D.19) for the functions $\widetilde{P}^m_l$ suggests to start with the direct sum of two representations of the form (5.3), such that both signs of the argument of $\widetilde{P}^m_l$ appear.

$$|l,m\rangle = \sum_{\sigma=\pm 1}\sum_{m_t} c_{l,m}^{m_t,\sigma} \; |m_t,m_k,\sigma\rangle \tag{6.5}$$

$$c_{l,m}^{m_t,\sigma} = \begin{cases} \sqrt{1-q^{-2}}q^{m_t-1-m}\widetilde{P}^m_l(\sigma q^{2(m_t-m-1})\text{ for} m \geq 0 \\ \sqrt{1-q^{-2}}q^{m_t-1}\widetilde{P}^{|m|}_l(\sigma q^{2(m_t-1)}) \quad \text{ for} m < 0 \end{cases} \tag{6.6}$$

We know that $m_k \geq 0$, $m_t \leq 0$ and $m = m_t + m_k$, thus m_t is restricted by $m_t \leq 0$ and $m \geq m_t$. The last condition comes into effect for negative values of m. Note that if m_t takes its largest allowed value, the coefficient of $c_{l,m}^{m_t+1}$ in (6.3) vanishes. We are free to choose this c to be zero. For $m \geq 0$ it then follows from (6.3) that $c_{l,m}^{m_t} = 0$ for $m_t > 0$ and for $m < 0$ the same is true for $m_t > m$.

The values of l are restricted by the condition $|m| \leq l$, as seen from (D.9). This is obviously consistent with the recursion formula (6.3).

We have chosen the normalization in (6.6) in such a way that according to (D.19) the eigenfunctions of ${\mathbf{T_{orb}}}^2$ are orthonormal:

$$\begin{aligned}&q^{-1}\lambda\sum_{\sigma=\pm 1}\sum_{m_t=-\infty}^{\min\{0,m\}} q^{2(m_t-1)-m-|m|}\widetilde{P}^{|m|}_l(\sigma q^{2(m_t-1)-m-|m|}) \\ &\times\widetilde{P}^{|m|}_{l'}(\sigma q^{2(m_t-1)-m-|m|}) = \delta_{l,l'}\end{aligned} \tag{6.7}$$

To see this for $m < 0$ it is enough to shift the summation variable $m_t \to m_t + m$.

We now assume that the two representations with $\sigma = +1$ and $\sigma = -1$ also lead to a different sign of z_0 in (5.6).

$$X^3 \; |m_t,m_k,M,\sigma\rangle = q^{2(m_t+M)}\sigma|z_0| \; |m_t,m_k,M,\sigma\rangle \tag{6.8}$$

Then it follows from (E.7) that the functions $\widetilde{P}^m_l$ satisfy the following completeness relation

$$\begin{aligned}&q^{-1}\lambda\sum_{l=0}^{\infty}q^{m_t+m_t'-2-m-|m|}\widetilde{P}^{|m|}_l(\sigma q^{2(m_t-1)-m-|m|}) \\ &\times\widetilde{P}^{|m|}_l(\sigma' q^{2(m_t'-1)-m-|m|}) = \delta_{\sigma,\sigma'}\delta_{m_t,m_t'}\end{aligned} \tag{6.9}$$

This construction shows that for fixed m, ${\mathbf{T_{orb}}}^2$ is a selfadjoint operator in the basis $|m_t,m_k,\sigma\rangle$, $\sigma = \pm 1$, and that the transformation from the basis $|m_t,m_k,\sigma\rangle$ to the basis $|l,m\rangle$ is an isometry.

Appendices

A Representations of the T algebra

When constructing representations of the T algebra, we are aiming at representations where T^3 is selfadjoint (or essentially selfadjoint). This allows us to assume T^3 to be diagonal:

$$T^3 \; |m\rangle = f(m) \; |m\rangle. \tag{A.1}$$

The eigenvalue of T^3 is $f(m)$, m is a labelling of the eigenstates.

The second equation of (2.3) shows that $T^+ \; |m\rangle$ is again an eigenstate of T^3, we choose the labelling such that this state is labelled by $m+1$:

$$T^+ \; |m\rangle = c_m \; |m+1\rangle. \tag{A.2}$$

The relation (2.3) leads to a recursion formula for $f(m)$:

$$f(m+1) = \frac{1}{q^4}f(m) + \frac{1}{q^2}(q+q^{-1}). \tag{A.3}$$

This recursion formula has the solution

$$f(m) = \frac{1}{\lambda} - dq^{-4m}. \tag{A.4}$$

From $\overline{T^3} = T^3$ follows that dq^{-4m} has to be real. We take d and m to be real.

For the operator τ of (2.10) follows

$$\tau \; |m\rangle = \lambda dq^{-4m} \; |m\rangle. \tag{A.5}$$

From the conjugation properties of T^+ it follows that

$$T^- \; |m\rangle = q^2 c^*_{m-1} \; |m-1\rangle. \tag{A.6}$$

The third equation of (2.3) is the conjugate of the second one. The first equation of (2.3) amounts to a recursion formula for $c_m^* c_m$

$$q c_{m-1}^* c_{m-1} - q^3 c_m^* c_m = f(m). \tag{A.7}$$

This recursion formula can be solved:

$$c_m^* c_m = \frac{1}{\lambda}\left\{ -\frac{1}{q^2\lambda} + \alpha\lambda q^{-2m} - \frac{d}{q^4} q^{-4m} \right\}. \tag{A.8}$$

The real parameter α is not determined by (A.7).

We see that $c_m^* c_m$ becomes negative for $m \to \infty$. This is not allowed. There has to be a largest value of m, say $\overline{m}$, such that

$$c_{\overline{m}}^* c_{\overline{m}} = 0. \tag{A.9}$$

Then it follows from (A.2) that T^+ does not lead to a state with a larger value than $\overline{m}$. To analyze this situation we introduce the function:

$$x = q^{-2m} \tag{A.10}$$

$$h(x) = \left\{ -\frac{1}{q(q^2-1)} + \alpha\lambda x - \frac{d}{q^4} x^2 \right\}$$

The function $h(x)$ is negative for $x = 0$, the sign of $h(x)$ for $x \to \infty$ depends on the sign of d. In any case $h(x)$ has to have a zero for positive x to represent $c^* c$. We have to demand

$$x_1 = q^{-2\overline{m}}, \qquad h(x_1) = 0. \tag{A.11}$$

The parameter α can now be expressed in terms of $\overline{m}$.

$$\alpha = \frac{1}{\lambda}\left\{ \frac{1}{\lambda} q^{2(\overline{m}-1)} + d q^{-2(\overline{m}+2)} \right\}. \tag{A.12}$$

If α takes this value $h(x)$ has the two zeros:

$$x_1 = q^{-2\overline{m}}, \qquad x_2 = \frac{1}{\lambda d} q^{2(\overline{m}+1)}. \tag{A.13}$$

We obtain for $c^* c$:

$$c_m^* c_m = -\frac{d}{\lambda q^4}\left(q^{-2m} - q^{-2\overline{m}}\right)\left(q^{-2m} - \frac{1}{\lambda d} q^{2(\overline{m}+1)}\right). \tag{A.14}$$

The representation is characterized by two parameters, $\overline{m}$ and d. We use this parameter in the explicit form of the matrix elements:

$$\begin{aligned}
T^3 \ |m\rangle &= \left(\frac{1}{\lambda} - d q^{-4m}\right) \ |m\rangle \\
T^+ \ |m\rangle &= \qquad\qquad\qquad\qquad\qquad\qquad (A.15) \\
&\frac{1}{q^2}\sqrt{\frac{d}{\lambda}\left(q^{-2m} - q^{-2\overline{m}}\right)\left(\frac{1}{\lambda d} q^{2(\overline{m}+1)} - q^{-2m}\right)} \ |m+1\rangle \\
T^- \ |m\rangle &= \sqrt{\frac{d}{\lambda}\left(q^{-2(m-1)} - q^{-2\overline{m}}\right)} \\
&\times\sqrt{\frac{1}{\lambda d} q^{2(\overline{m}+1)} - q^{-2(m-1)}} \ |m-1\rangle \\
\tau \ |m\rangle &= d\lambda q^{-4m} \ |m\rangle \\
\mathbf{T}^2 &= \frac{1}{\lambda^2\sqrt{\lambda d}}\left(q^{2(\overline{m}+1)} + \lambda d q^{-2\overline{m}}\right) - \frac{1+q^2}{\lambda^2}
\end{aligned}$$

Let us now have a closer look at the condition $c^* c \geq 0$. For this purpose we discuss the three cases $d > 0$, $d = 0$ and $d < 0$ separately.

$d > 0$

There has to be a smallest value of m, say $\underline{m}$, such that $|\underline{m}\rangle \neq 0$ and $T^- \ |\underline{m}\rangle = 0$, therefore

$$c_{\underline{m}-1}^* c_{\underline{m}-1} = 0. \tag{A.16}$$

From (A.14) follows for $\overline{m} \geq 0$

$$d = \frac{1}{\lambda}, \qquad \underline{m} = -\overline{m}. \tag{A.17}$$

The number of states between $\overline{m}$ and $\underline{m}$ has to be integer:

$$2\overline{m} + 1 \equiv n. \tag{A.18}$$

This shows that $\overline{m}$ has to be integer or half integer and we found the $2l+1$-dimensional representation ($\overline{m} = l$) of $so_q(3)$.

$d = 0$

In this case $h(x)$ is a linear function:

$$h(x) = -\frac{1}{q(q^2-1)} + \alpha\lambda x. \tag{A.19}$$

Now α has to be positive for h to have a zero for positive x. From (A.12) follows

$$\alpha = \frac{1}{q^2\lambda^2} q^{2\overline{m}}. \tag{A.20}$$

The representation can be obtained from (A.15). The parameter $\overline{m}$ that characterizes the representation can take any real value. The representation is infinite-dimensional, however, τ is not invertible.

$d < 0$

This is the situation that arises for the t algebra, as can be seen from (3.2). In this case x_2 is negative. We only have a largest value of m. The representation is infinite-dimensional and $\overline{m}$ is not restricted. The matrix elements are obtained from (A.15). We write them such as to exhibit the positive square roots:

$$\begin{aligned}
T^3 \ |m\rangle &= \left(\frac{1}{\lambda} - d q^{-4m}\right) \ |m\rangle \\
T^+ \ |m\rangle &= \qquad\qquad\qquad\qquad\qquad\qquad (A.21) \\
&\frac{1}{q^2}\sqrt{-\frac{d}{\lambda}}\sqrt{\left(q^{-2m} - q^{-2\overline{m}}\right)\left(q^{-2m} - \frac{1}{\lambda d} q^{2(\overline{m}+1)}\right)} \ |m+1\rangle \\
T^- \ |m\rangle &= \sqrt{-\frac{d}{\lambda}}\sqrt{\left(q^{-2(m-1)} - q^{-2\overline{m}}\right)} \\
&\times\sqrt{q^{-2(m-1)} - \frac{1}{\lambda d} q^{2(\overline{m}+1)}} \ |m-1\rangle \\
\tau \ |m\rangle &= d\lambda q^{-4m} \ |m\rangle
\end{aligned}$$

τ has negative eigenvalues only, thus $\tau^{\frac{1}{2}}$ and $\mathbf{T}^2$ will not be real.

B Representations of the K algebra

The algebraic relations of the K algebra are the same as the relations of the T algebra, they are different only as a $*$ algebra:

$$\overline{K^3} = K^3, \quad \overline{K^+} = -\frac{1}{q^2}K^- \tag{B.1}$$
$$\overline{T^3} = T^3, \quad \overline{T^+} = \frac{1}{q^2}T^-$$

This makes the K algebra a $su_q(1,1)$ algebra.

All the results that depend only on the algebraic relations are the same as for the T algebra.

$$(A.1): K^3 \;|m\rangle = \phi(m)\;|m\rangle \tag{B.2}$$
$$(A.2): K^+ \;|m\rangle = \gamma_m \;|m\rangle \tag{B.3}$$
$$(A.4): \quad \phi(m) = \frac{1}{\lambda} - d_k q^{-4m} \tag{B.4}$$

We again take d_k and m real.

$$(A.5): \tau_k \;|m\rangle = \lambda d_k q^{-4m}\;|m\rangle \tag{B.5}$$

For K^- there is a change in sign due to (B.1):

$$(A.6): \; K^- \;|m\rangle = -q^2\gamma^*_{m-1}\;|m-1\rangle \tag{B.6}$$
$$(A.8): \; \gamma^*_m\gamma_m = -\frac{1}{\lambda}\left\{-\frac{1}{q^2\lambda} + \alpha\lambda q^{-2m} - \frac{d_k}{q^4}q^{-4m}\right\} \tag{B.7}$$

Now $\gamma^*_m\gamma_m$ becomes positive for $m \to \infty$, we do not have to cut off the spectrum at a largest value of m. We shall see that all the representations are infinite-dimensional.

We introduce the function $\kappa(x)$ analogous to $h(x)$ in (A.10):

$$\kappa(x) = \left\{\frac{1}{q^2\lambda^2} - \alpha x + \frac{d_k}{\lambda q^4}x^2\right\}, \qquad x = q^{-2m}. \tag{B.8}$$

The representations of the K algebra are:

$$\begin{aligned} K^3 \;|m\rangle &= (\frac{1}{\lambda} - d_k q^{-4m})\;|m\rangle \\ K^+ \;|m\rangle &= \sqrt{\kappa(q^{-2m})}\;|m+1\rangle \\ K^- \;|m\rangle &= -q^2\sqrt{\kappa(q^{-2(m-1)})}\;|m-1\rangle \end{aligned} \tag{B.9}$$

They are characterized by α and d_k and restricted by the condition $\kappa(q^{-2m}) \geq 0$. To discuss this condition we determine the zeros of $\kappa(x)$

$$\begin{aligned} &\kappa(x_{1,2}) = 0 \\ &x_{1,2} = \frac{\lambda}{2d_k q^{-4}}\left\{\alpha \pm \sqrt{\alpha^2 - 4d_k q^{-6}\lambda^{-3}}\right\} \end{aligned} \tag{B.10}$$

We discuss the cases $d_k > 0$, $d_k = 0$ and $d_k < 0$ separately and start with

$d_k > 0$

In this case $\kappa(x)$ has no positive zero for $\alpha < 2q^{-3}\sqrt{d_k\lambda^{-3}} = \alpha_0$. The range of m is not restricted, it can be of the form $m_0 + n$, $n \in \mathbb{Z}_0$. If $\alpha \geq \alpha_0$ we will have two positive zeros and $\kappa(x)$ can be written in the form

$$\kappa(x) = \frac{1}{q^2\lambda^2 x_1 x_2}(x - q^{-2\overline{m}})(x - q^{-2\underline{m}}). \tag{B.11}$$

The values of the zeros x_1, x_2 determine the parameters α and d_k and therefore the representation.

$$\begin{aligned} d_k &= \frac{q^2}{\lambda}q^{2(\overline{m}+\underline{m})} \\ \alpha &= \frac{1}{q^2\lambda^2}(q^{2\underline{m}} + q^{2\overline{m}}) \end{aligned} \tag{B.12}$$

There are now two inequivalent representations. We find that in one representation the allowed values of m are

$$m \leq \overline{m}, \qquad m = \overline{m}, \overline{m}-1, \overline{m}-2, \ldots \tag{B.13}$$

For the other representation we find

$$m > \underline{m}, \qquad m = \underline{m}+1, \underline{m}+2, \ldots \tag{B.14}$$

Now we consider

$d_k = 0$

The function κ becomes linear. It is positive at $x = 0$ and, depending on α, stays positive or becomes negative for $x \to \infty$. If $\alpha < 0$ there is no restriction in the range of m, $m = m_0 + n$, $n \in \mathbb{Z}$. If $\alpha > 0$ there is a lowest eigenvalue of m, we are at the situation of (B.14).

Finally we consider

$d_k < 0$

In this case $\kappa(x)$ is positive for $x \to 0$ and negative for $x \to \infty$. There is one zero for $x > 0$. This can also be seen from (B.10) because the square root will now be larger than α. The relevant zero of $\kappa(x)$ is:

$$\begin{aligned} x_1 &= \frac{\lambda}{2|d_k|q^{-4}}\left\{-\alpha + \sqrt{\alpha^2 + 4|d_k|q^{-6}\lambda^{-3}}\right\} \\ &= q^{-2\underline{m}_k} \end{aligned} \tag{B.15}$$

Now all values of α are allowed. The range of m will be as in (B.14).

For orbital angular momentum we encounter the representation

$$d_k = -\frac{1}{q^2\lambda}, \qquad x_1 = q^2. \tag{B.16}$$

This leads to $\alpha = 0$ and

$$\kappa(q^{-2m}) = \frac{1}{\lambda^2 q^2}\left(1 - q^{-4(m+1)}\right). \tag{B.17}$$

The respective representation is shown in (5.2).

C Comultiplication

The standard comultiplication rule for the algebra (2.3) is:

$$\begin{aligned} \Delta(T^3) &= T^3 \otimes 1 + \tau \otimes T^3 \\ \Delta(T^\pm) &= T^\pm \otimes 1 + \tau^{\frac{1}{2}} \otimes T^\pm \end{aligned} \tag{C.1}$$

As a consequence, τ is group-like:

$$\Delta(\tau) = \tau \otimes \tau. \tag{C.2}$$

The algebra (2.3) is the same for the T algebra and the K algebra, they are distinguished by their conjugation properties (B.1).

As long as $\tau^{\frac{1}{2}}$ is hermitean, (C.1) will respect the conjugation properties and we have a comultiplication within the T algebra or the K algebra respectively. From (C.2) follows that $\Delta(\tau^{\frac{1}{2}})$ will be hermitean if $\tau^{\frac{1}{2}}$ is.

$$\Delta(\tau^{\frac{1}{2}}) = \tau^{\frac{1}{2}} \otimes \tau^{\frac{1}{2}} \tag{C.3}$$

If $\tau^{\frac{1}{2}}$ is not hermitean $\Delta(T)$ will have no definite conjugation properties even if T has.

We now turn to the product of representations as it follows from the comultiplication rule (C.1). If we have two representations of the algebra (2.3) we obtain a new one by the rule

$$\begin{aligned} \Delta(T^3) &= T_1^3 \otimes 1 + \tau_1 \otimes T_2^3 \\ \Delta(T^\pm) &= T_1^\pm \otimes 1 + \tau_1^{\frac{1}{2}} \otimes T_2^\pm \end{aligned} \tag{C.4}$$

From the discussion above follows that we can multiply two representations of the T algebra (K algebra) to obtain a representation of the T algebra (K algebra) as long as $\tau_1^{\frac{1}{2}}$ is hermitean. From now on we shall drop the indices 1 and 2 again, first and second representations will be defined by the position in the product (C.4).

That the $\tau^{\frac{1}{2}}$ of the first representation is hermitean means $d_1 > 0$. We shall discuss this situation first.

$d_1 > 0$

The product of two representations of the T algebra (K algebra) will be a T algebra (K algebra). From (C.2) follows

$$d = \lambda d_1 d_2. \tag{C.5}$$

If d_2 is negative d will be negative as well.

For the T algebra d positive restricts d to be $d = \frac{1}{\lambda}$. This characterizes the finite-dimensional representations. From (C.5) follows that the product of two finite-dimensional representations is finite-dimensional as expected but also that the product of a finite-dimensional representation ($d_1 = \frac{1}{\lambda}$) with an infinite-dimensional representation ($d_2 < 0$) leads to $d < 0$ and cannot be reduced to finite-dimensional representations.

For the K algebra all representations are infinite-dimensional.

We now turn to the case that d_1 is negative, $\tau_1^{\frac{1}{2}}$ will be anti-hermitean.

$d_1 < 0$

In this case the product of two representations will in general not have well-defined conjugation properties. We can, however, start from a modified comultiplication rule:

$$\begin{aligned} \Delta_\beta(T^3) &= T^3 \otimes 1 + \tau \otimes T^3 \\ \Delta_\beta(T^\pm) &= T^\pm \otimes 1 \pm (-\tau)^{\frac{1}{2}} \otimes T^\pm \end{aligned} \tag{C.6}$$

If $(-\tau)^{\frac{1}{2}}$ is hermitean this rule allows us to multiply a representation of the $T(K)$ algebra by a representation of the $K(T)$ algebra to obtain a $T(K)$ algebra.

$$\begin{aligned} &T^3 \otimes 1 + \tau \otimes K^3 \\ &T^\pm \otimes 1 \pm (-\tau)^{\frac{1}{2}} \otimes K^\pm \end{aligned} \tag{C.7}$$

will be a representation of the T algebra whereas

$$\begin{aligned} &K^3 \otimes 1 + \tau_k \otimes T^3 \\ &K^\pm \otimes 1 \pm (-\tau_k)^{\frac{1}{2}} \otimes T^\pm \end{aligned} \tag{C.8}$$

will be a representation of the K algebra.

For the comultiplication (C.6) τ will be group-like as well and it follows again that

$$d = \lambda d_1 d_2. \tag{C.9}$$

But now d_1 is negative.

Of special interest is the case that d_1 and d_2 are both negative, then d is positive. If we multiply $T \times K$ to obtain a T algebra then we know that $d = \frac{1}{\lambda}$ and, as a consequence

$$d_1 d_2 = \frac{1}{\lambda^2} \tag{C.10}$$

to obtain a representation with well-defined conjugation properties. This is exactly the case for the construction of the T_{orb} algebra in the main part of this paper.

D The big q-Jacobi polynomials

In this appendix we recall some basics about q-special functions [8], [9], [10], in particular the big q-Jacobi polynomials.

First, we introduce some useful notation. The expressions

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} \overset{q \to 1}{\to} a, \quad [a]! = \prod_{k=1}^{a} [k] \overset{q \to 1}{\to} a! \tag{D.1}$$

are known as symmetric q-numbers and symmetric q-factorials respectively. The corresponding q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[k]![n-k]!} & \text{for} n \geq k, \\ 0 & \text{for } n < k \text{ or} n, k < 0. \end{cases} \tag{D.2}$$

Of course $\begin{bmatrix} n \\ k \end{bmatrix} \stackrel{q\to 1}{\to} \begin{pmatrix} n \\ k \end{pmatrix}$. There are also "unsymmetric" counterparts of these objects: the basic q-number

$$\frac{1-q^a}{1-q} \stackrel{q\to 1}{\to} a \tag{D.3}$$

and the q-shifted factorial (Pochammer-symbol)

$$(a;q)_k = \prod_{n=0}^{k-1}(1-aq^n), \tag{D.4}$$

$$(a_1, a_2, \ldots, a_i; q)_k = \prod_{m=1}^{i}(a_m;q)_k.$$

The Jackson integral of a function $f(x)$ is defined for $q>1$ by

$$\int_0^a d_{q^{-1}}x\, f(x) = (1-q^{-1})\sum_{\nu=0}^{\infty} aq^{-\nu} f(aq^{-\nu}) \tag{D.5}$$

With the help of the q-shifted factorials, the basic hypergeometric function can be introduced

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix}\middle| q^{-1};x\right) = \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q^{-1})_k}{(b_1,\ldots,b_s;q^{-1})_k} \times(-1)^{(1+s-r)k}q^{-\frac{1}{2}(1+s-r)k(k-1)}\frac{x^k}{(q^{-1};q^{-1})_k} \tag{D.6}$$

This series plays in the theory of q-special functions a role analogous to that of the hypergeometric series in the theory of usual special functions. We have considered a base q^{-1} here, because in this way ${}_r\phi_s$ is well-defined for $q>1$, which is the case we are interested in here.

The big q-Jacobi polynomials [8], [9] are constructed in terms of the basic hypergeometric series as

$$P_l(x;a,b,c;q^{-1}) = {}_3\phi_2\left(\begin{matrix} q^l, abq^{-(l+1)}, x \\ aq^{-1}, cq^{-1}\end{matrix}\middle| q^{-1};q^{-1}\right). \tag{D.7}$$

For the applications we consider in this paper, we are interested in the case

$$\begin{aligned} P_l^m(x) &\equiv P_{l-m}(x;q^{-2m},q^{-2m},-q^{-2m};q^{-2}) \\ &= \sum_{k=0}^{l-m}(-1)^k\frac{q^{-k(m+1)}(x;q^{-2})_k}{(-q^{-2(m+1)};q^{-2})_k}\begin{bmatrix} l-m \\ k\end{bmatrix}\begin{bmatrix} l+m+k \\ k\end{bmatrix} \\ &\times\begin{bmatrix} m+k \\ k\end{bmatrix}^{-1}, \qquad m\ge 0 \end{aligned} \tag{D.8}$$

Notice that the P_l^m are polynomials of the order $l-m$ in x. Due to the factor $\begin{bmatrix} l-m \\ k\end{bmatrix}$, which vanishes for $k>l-m$ according to the definition (D.2). The sum in (D.6) actually becomes finite. Due to the same factor the polynomials P_l^m vanish, if the condition $m\le l$ is not satisfied:

$$P_l^m(x) = 0 \quad \text{for} l<m. \tag{D.9}$$

The further condition $m\ge 0$ is necessary for the polynomials P_m^l to be well-defined, due to the factor $(q^{-2(m-1)};q^{-2})_k$ in the denominator of the basic hypergeometric function which otherwise vanishes for negative m.

Some of the first big q-Jacobi polynomials are

$$\begin{aligned} &P_0^0(x) = 1, && P_1^0(x) = x, \\ &P_2^0(x) = \frac{1}{q[2]}([3]x^2-q^{-2}), && P_3^0(x) = \frac{x}{q^5[2]}([5]q^2x^2-[3]), \\ &P_1^1(x) = 1, && P_2^1(x) = x, \\ &P_3^1(x) = \frac{1}{q^5[4]}(q^4[5]x^2-1). \end{aligned} \tag{D.10}$$

From [8] and [9] we learn that the polynomials $P_l^m(x)$ satisfy a recurrence relation

$$\begin{aligned} xq^m[2l+1]P_l^m(x) &= \\ q^l[l+m+1]P_{l+1}^m(x) &+ q^{-l-1}[l-m]P_{l-1}^m(x) \end{aligned} \tag{D.11}$$

a q-difference equation

$$\begin{aligned} &\left(q^{-1-2m}(q^{2l+1}+q^{-2l-1})x^2 - q^{-4(m+1)}(q^2+1)\right)P_l^m(x) \\ &\quad = q^{-2(2m+1)}(x^2-1)P_l^m(xq^{-2}+(x^2) \\ &\qquad -q^{-4(m+1)})P_l^m(xq^2) \end{aligned} \tag{D.12}$$

and the orthonormality condition

$$\int_{-q^{-2(m+1)}}^{q^{-2(m+1)}} d_{q^{-2}}x\; w_l^m(x)w_{l'}^m(x)P_l^m(x)P_{l'}^m(x) = \delta_{l,l'}. \tag{D.13}$$

Here, the weight function w_l^m is defined by

$$\begin{aligned} w_l^m(x) \equiv &\sqrt{\frac{\left(q^{-4(m+1)};q^{-4}\right)_\infty}{\left(q^{-4},q^{-4(m+1)-2};q^{-4}\right)_\infty\left(-q^{-2};q^{-2}\right)_\infty}} \\ &\times q^{-\frac{1}{2}(l^2+l+2lm-3m^2+m+3)}\sqrt{\left(x^2q^{4m};q^{-4}\right)_m} \\ &\times\sqrt{\frac{(q^{-2};q^{-2})_{l-m}}{(q^{-2(2m+1)};q^{-2})_{l-m}}}\sqrt{\frac{[2m+1]}{2[2l+1]}}. \end{aligned} \tag{D.14}$$

Actually, as it is done e.g. in [9], the big q-Jacobi polynomials can be alternatively defined as those polynomials in x which are orthonormal with respect to the Jackson integral with the weight function $w_l^m(x)$ in (D.14).

The weight function has the following scaling properties

$$w_l^m(xq^{-2}) = w_l^m(x)\sqrt{\frac{(1-x^2)}{(1-x^2q^{4m})}}, \tag{D.15}$$

$$w_{l-1}^m(x) = w_l^m(x)q^l\sqrt{\frac{[l+m][2l+1]}{[l-m][2l-1]}}.$$

It is useful for the purposes of this paper to absorb the weight function in the definition of the polynomials themselves and to introduce the further notation

$$\widetilde{P}_l^m(x) = w_l^m(x)P_l^m(x). \tag{D.16}$$

With the help of (D.15) it turns out that (D.11) and (D.12) become respectively

$$x\, q^{m+1}\widetilde{P}_l^m(x) = \sqrt{\frac{[l-m+1][l+m+1]}{[2l+1][2l+3]}}\ \widetilde{P}_{l+1}^m(x) + \sqrt{\frac{[l+m][l-m]}{[2l+1][2l-1]}}\ \widetilde{P}_{l-1}^m(x) \quad \text{(D.17)}$$

and

$$\left((q^{2l+1}+q^{-2l-1})\frac{x^2}{q} - (q^2+1)q^{-2(m+2)}\right)\widetilde{P}_l^m(x) = q^{-2(m+1)}\sqrt{(x^2-1)(x^2q^{4m}-1)}\ \widetilde{P}_l^m(xq^{-2}) + \sqrt{(x^2-q^{-4(m+1)+1})(x^2-q^{-4})}\ \widetilde{P}_l^m(xq^2) \quad \text{(D.18)}$$

By using (D.13) and the definition of the Jackson integral (D.5) we obtain the following orthonormality condition for the functions $\widetilde{P}_l^m(x)$

$$\sum_{\sigma=\pm1}\sum_{n=-\infty}^{0} q^{2(n-m-1)}\widetilde{P}_l^m(\sigma q^{2(n-m-1)})\widetilde{P}_{l'}^m(\sigma q^{2(n-m-1)}) = (1-q^{-2})^{-1}\delta_{l,l'}. \quad \text{(D.19)}$$

Moreover, the functions $\widetilde{P}_l^m(x)$ have the property that they transform under a parity transformation like

$$\widetilde{P}_l^m(-x) = (-1)^{l-m}\widetilde{P}_l^m(x). \quad \text{(D.20)}$$

In the particular case $m=0$ the big q-Jacobi polynomials become the big q-Legendre polynomials, which in the limit $q\to 1$ yield the usual Legendre polynomials. In the same limit from the polynomials $P_l^m(x)$ we recover the Jacobi polynomials with the normalization $P_l^m(1)=1$.

E Diagonalization of X^3

In this appendix we study the transformation which is inverse to the transformation (6.5), (6.6) constructed in Sect. 5. We show how the big q-Jacobi polynomials can be used to diagonalize X^3 in the basis where $\mathbf{T_{orb}}^2$, T^3_{orb}, R^2 are diagonal.

The representation where $\mathbf{T_{orb}}^2$, T^3_{orb}, R^2 are diagonal can be found in [11], [12], [13]

$$\begin{aligned}
\mathbf{T}^2_{orb}\ |M,l,m\rangle &= q[l][l+1]\ |M,l,m\rangle \\
X^3\ |M,l,m\rangle &= r_0 q^{2M+m} \\
&\times\left\{\sqrt{\frac{[l+m+1][l-m+1]}{[2l+1][2l+3]}}\ |M,l+1,m\rangle + \sqrt{\frac{[l+m][l-m]}{[2l+1][2l-1]}}\ |M,l-1,m\rangle\right\} \quad \text{(E.1)}\\
X^+\ |M,l,m\rangle &= r_0 q^{2M+m} \\
&\times\left\{q^{-l}\sqrt{\frac{[l+m+1][l+m+2]}{[2][2l+1][2l+3]}}\ |M,l+1,m+1\rangle - q^{l+1}\sqrt{\frac{[l-m][l-m-1]}{[2][2l+1][2l-1]}}\ |M,l-1,m+1\rangle\right\}\\
X^-\ |M,l,m\rangle &= r_0 q^{2M+m} \\
&\times\left\{q^{l}\sqrt{\frac{[l-m+1][l-m+2]}{[2][2l+1][2l+3]}}\ |M,l+1,m-1\rangle - q^{-l-1}\sqrt{\frac{[l+m][l+m-1]}{[2][2l+1][2l-1]}}\ |M,l-1,m-1\rangle\right\}
\end{aligned}$$

where

$$0\le l<\infty, \qquad -l\le m\le l. \quad \text{(E.2)}$$

We make the following Ansatz for an eigenfunction of X^3

$$X^3\sum_{M,l,m} d_{M,l,m}|M,l,m\rangle = z\sum_{M,l,m} d_{M,l,m}|M,l,m\rangle, \quad \text{(E.3)}$$

with z the corresponding eigenvalue. By using (E.2) we obtain a recursion relation for the coefficients $d_{M,l,m}$

$$z\ d_{M,l,m} = r_0\frac{q^{2M+m}}{\sqrt{[2l+1]}}\times\left\{\sqrt{\frac{[l-m+1][l+m+1]}{[2l+3]}}d_{M,l+1,m} + \sqrt{\frac{[l+m][l-m]}{[2l-1]}}d_{M,l-1,m}\right\}. \quad \text{(E.4)}$$

A comparison with the recursion relation (D.17) for the functions $\widetilde{P}_l^m$ defined in (D.16) in terms of the Jacobi polynomials shows that a solution of (E.4) is

$$d^{\nu,\sigma}_{M,l,m} = \begin{cases}\dfrac{\sqrt{1-q^{-2}}}{q^{M+1+m-\nu}}\widetilde{P}_l^m(\sigma q^{2(\nu-M-1-m)}) & \text{for} m\ge 0\\[2ex] \dfrac{\sqrt{1-q^{-2}}}{q^{M+1-\nu}}\widetilde{P}_l^{|m|}(\sigma q^{2(\nu-M-1)}) & \text{for} m<0\end{cases} \quad \text{(E.5)}$$

where $z=\sigma r_0 q^{-1+2\nu}$, $\sigma=\pm1$. By comparing with the form of the eigenvalues of X^3 (5.8) we see that we have to restrict

$$\nu, M\in\mathbb{Z}, \quad \nu\le M, \quad m\ge\nu-M. \quad \text{(E.6)}$$

Notice that the argument of the functions would correspond to $x=\cos\theta=\frac{z}{r}$ classically, apart from the q-factor $q^{-(m+|m|)}\to 1$ for $q\to 1$.

X^3 is a self-adjoint operator in this representation. This was shown in [14].

Now, the set of eigenfunctions of a self-adjoint operator is complete, therefore we expect a completeness relation to hold for the eigenfunctions of X^3. In fact, (D.19) can be interpreted in this way. As the sum (D.19) contains two sums, one where the argument of P^l_m is positive and one where it is negative, we obtain a representation where the eigenvalues of X^3 can have both signs, so that we automatically find the direct sum of two representations of the type (5.8). The normalization of the coefficients $d_{M,l,m}$ in (E.5) has been chosen in such a way as to yield exactly (D.19).

As the eigenfunctions of a selfadjoint operator corresponding to different eigenvalues are orthogonal, since the normalization constant is already fixed by (D.19), we argue that the following relation holds

$$\sum_{l=0}^{\infty} q^{\nu+\nu'-2}\widetilde{P}_l^{|m|}(\sigma q^{2(\nu-1)})\widetilde{P}_l^{|m|}(\sigma' q^{2(\nu'-1)}) = (1-q^{-2})^{-1}\delta_{\nu,\nu'}\delta_{\sigma,\sigma'} \tag{E.7}$$

where $\sigma, \sigma' = \pm 1$ are the signs of the argument of the functions and $\nu \leq \min\{m, 0\}$. This is an interesting result for itself about the Jacobi polynomials.

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